# A practical introduction to the kinetic simulation of plasmas

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## Introduction

This practical is an introduction to the kinetic simulation of plasmas. Throughout this work, you will use the open-source Particle-In-Cell (PIC) code Smilei [1] to address various physical mechanisms that have a particular importance in plasma physics.

The practical is structured in **4 projects**:
- the first two projects focus on plasma instabilities driven by counter-streaming cold (electron) plasmas. The modelling of both processes can be in that case (cold plasmas) performed in the framework of a relativistic cold fluid model. Details on this modelling are given in the *supplemental material* at the end of this document.
- the two last projects will on the other end focus on purely kinetic processes in the sense that they can only be accounted for in the framework of the kinetic theory of plasmas. These two processes will be related to the kinetic behaviour of Langmuir waves (a.k.a. electron plasma waves).

**Nota bene** – You may find this introduction quite evasive. This is done on purpose! Indeed, we wish you to discover by yourself, running and analysing the simulations, what physical processes are at play in the different projects. Hence, as you will see, all 4 projects have no title and it will be up to you to give them one.

## A quick word on Smilei

As previously stated, the numerical tool you will use for this practical is the PIC code Smilei. It is an open-source and collaborative code freely distributed under a CeCILL-B license (equivalent to the GPL license for free-softwares). The code, its documentation and post-processing tools are freely available on SMILEI's website hosted on GitHub: `https://smileipic.github.io/Smilei/index.html`.

The focus of this practical is on the physics of plasmas, not on how to use Smilei. Hence, a prior knowledge of Smilei is not mandatory. Yet, checking Smilei's website for information on how to write a *namelist* can be useful. Furthermore, the interested reader can find additional tutorials accessible from the code's website, and that focus on other physical processes and/or on how to use Smilei.

Last, all simulations presented in this practical will be run in 1D3V geometry in order to run in a short time over a single CPU. 1D3V means that only one dimension in space is considered, but particles move in a three-dimensional velocity space (that is the particle velocity is a three-dimensional vector). This is mandatory to address electromagnetic problems. Note however that the version of Smilei you have is the full research code (not a downgraded version!), hence, it can address more evolved problems in higher dimensions.

## Normalizations

SMILEI is an electromagnetic PIC code, that is, it solves the Maxwell-Vlasov system of equations that describes the evolution of various species of a collisionless plasma in the self-consistent electromagnetic fields. When dealing with this system of equations, it is interesting to introduce the normalizations given in Table 1. In this work, all quantities given to the code, as well as all quantities provided by the code as outputs will be in normalized units.

| | |
|---|---|
| **Units of time** | $\omega_r^{-1}$ |
| Units of velocity | $c$ |
| Units of charge | $e$ |
| Units of mass | $m_e$ |
| Units of momentum | $m_e\, c$ |
| Units of energy, temperature | $m_e c^2$ |
| Units of length | $c/\omega_r$ |
| Units of number density | $n_r = \epsilon_0\, m_e\, \omega_r^2/e^2$ |
| Units of current density | $e\, c\, n_r$ |
| Units of pressure | $m_e\, c^2\, n_r$ |
| Units of electric field | $m_e\, c\, \omega_r/e$ |
| Units of magnetic field | $m_e\, \omega_r/e$ |
| Units of Poynting flux | $m_e\, c^3\, n_r/2$ |

Table 1: List of the most common normalizations used in SMILEI.

As shown in Table 1, all charges and mass will be normalized to the elementary charge $e$ and electron mass $m_e$, respectively. Furthermore, all velocities will be normalized to the speed of light in vacuum $c$ that naturally appears from Maxwell's equations. Now, the unit of time - here defined as $\omega_r^{-1}$, with $\omega_r$ the reference angular frequency - is not defined a priori, and is chosen by the user. Once this unit of time is chosen, all other units are uniquely defined and follow as detailed in Table 1. Note however that number densities associated to the plasma species are not in units of $(\omega_r/c)^3$ but in units of $n_r = \epsilon_0\, m_e\, \omega_r^2/e^2$.

**Exercise** – In this work, we will consider plasmas with a well-defined mean density $n_0$. In that case, it is convenient to normalise times to the *electron plasma frequency* at this density $\omega_{p0} = e^2 n_0/(\epsilon_0 m_e)$. What will be the units of length, density, electric and magnetic fields?

*Optional question* – If one were considering the interaction of an electromagnetic wave with angular frequency $\omega_0$ with a plasma with density $n_0$, one may however prefer to use $\omega_0$ as reference angular frequency. In that case, what would be the reference density $n_r$? What does it correspond to? Similarly, what would be the reference for the electric field, and what would that field correspond to?

# Project 1

## Simulation set-up

We here consider a plasma with density $n_0$ made of immobile ions, and two counter-streaming electron flows, each with density $n_0/2$ and opposite velocities $\pm v_0\,\hat{\mathbf{x}}$, with $\hat{\mathbf{x}}$ the direction resolved in this 1D simulation (note that throughout this work, we will consider 1D3V simulations with a single dimension in space, but particle velocities having all three components).

All the simulation (input) information are given in the input file `project1.py`. Smilei input files are written in Python. You can define as many parameters as you want, then feed Smilei's input blocks such as the `Main()` or `Species()` blocks.

**1.a** – Open `project1.py` with your favorite text editor. Start by choosing `grid_length=1.68`, which defines the simulation grid length to $1.68c/\omega_{pe}$. Have a look at the `electron1` and `electron2` `Species()` blocks. What does setting `grid_length=1.68` mean in terms of the initial simulation set-up?

## First run

Now, we prepare to run the simulation.
- If you are using the virtual machine, you just need to run the command `run_smilei project1`. This will (i) create a directory `project1/` where all the input/output files will be copied/written, (ii) launch the code, then (iii) open a graphic interface to analyse/plot the (output) data.
- If you have installed Smilei yourself on your machine, it is convenient to create a new directory, let's call it `project1` in which you copy the input file `project1.py`. You will then run the code, e.g. using `./smilei project1.py`. To access the data, you will then use the graphic interface which, if it is not yet installed, you can find it here: `https://drive.google.com/drive/folders/1oIi7zerdWtFIRFRbhuY6LtxRjlydAi7q?usp=sharing`.

Once the code has run, you need to analyse the data using the graphic interface. Remember that all quantities are in normalized units!

**1.b** – First check the $E_x$ and $\rho$ (charge density) fields at time $t = 0$. Are they what you expect them to be? Actually, where does $E_x$ comes from (note that we do not specify any $E_x$ field at time $t = 0$)? Then, check the energy in the different electromagnetic fields (given as a function of time), e.g. `Uelm_Ex` denotes the total energy in the $E_x$-component of the electric field. In which field is the energy stored? For this field, how is the energy evolving with time? What does this mean: is this set-up stable or does it lead to an instability? Where does the energy in the electric field comes from?

**1.c** – Knowing which energy is growing, what is the nature of the physics at play? Checking the *supplemental material*, can you give a name to what you are observing?
*Additional question* – In the *supplemental material*, this instability was described coupling (cold) fluid equations with Maxwell's equations. Could I have used a different set of equations to describe the instability?

**1.d** – Add to your visualisation the diagnostics on the electric field $E_x$ and the $x - v_x$ phase-space diagnostics. Click on play, and have a look at what's going on in your simulation. Can you figure out the linear and non-linear stage of the instability?

**1.e** – In the linear stage of the instability, can you extract the growth rate of the instability? How does it compare to the theory?

**1.f** – Now, checking the $x - v_x$ phase-space at the moment of saturation (end of the linear stage), can you infer what leads to the saturation of this instability?

## Additional runs varying the seeded wavenumber

We will now rerun the same simulation set-up but changing the `grid_length` parameter, that is changing the wavenumber $k$ of the seeded mode.

**1.g** – Run the simulation with `grid_length=1.03`. What growth rate do you get for the instability? How does it compare with the theory? Do the same setting `grid_length=0.69`. Has the physics changed in the last 3 runs?

**1.h** – Run the simulation with `grid_length=0.62`. What has changed? Can you compare your simulation results with the theory? What are now the physical quantities that can be extracted from the simulation and compared to the theory? Continue decreasing the seeded wavelength, using `grid_length=0.31` then `grid_length=0.16` and checking the code predictions with the theory presented in the *supplemental material*.

## Physical interpretation

**1.i** – Can you give a simple, physical interpretation to what is going on?

# Project 2

## Simulation set-up

We now consider a set-up quite similar to the one previously investigated: a plasma with density $n_0$ made of immobile ions, and two counter-streaming electron flows, each with density $n_0/2$ and opposite velocities $\pm\mathbf{v_0}$. Now, however, the flow velocity are oriented along the $\hat{\mathbf{z}}$-direction, not resolved in this 1D simulation. In addition, a small perturbation in the $B_y$ field is seeded at the beginning of the simulation (time $t = 0$).

## First run

Start by running SMILEI using the `project2.py` input file.

**2.a** – Plot the $E_x$, $B_y$ fields as well as the electron species densities $\rho_{e,1}$ and $\rho_{e,2}$. Is it consistent with what you expect? In contrast with **project 1**, how is the system perturbed?

**2.b** – Then, check the energy in the different electromagnetic fields (given as a function of time). In which fields is the energy stored? For these fields, how is the energy evolving with time? What does this mean: is this set-up stable or does it lead to an instability? Where does the energy in the electromagnetic field comes from?

**2.c** – Knowing which energy is growing, what is the nature of the physics at play? Checking the *supplemental material*, can you give a name to what you are observing?

**2.d** – Add to your visualisation the diagnostics on the electric field $E_x$ and the $x - v_x$ phase-space diagnostics. Click on play, and have a look at what's going on in your simulation. Can you figure out the linear and non-linear stage of the instability?

**2.e** – In the linear stage of the instability, compute the growth rate associated to the $E_x$, $E_z$ and $B_y$ fields. How do the growth rate compare? What does this mean in terms of the linear phase of the instability, and why is $E_x$ behaving differently? What is the growth rate of the instability and how does it compare to the theory?

## Additional runs varying the seeded wavenumber

We will now rerun the same simulation set-up but changing the seeded wave number $k$ (variable `k` in the input file). Run successively the simulation with `k=1`, `k=0.5`, `k=0.2` and `k=0.1`. For all cases, extract the growth rate of the instability and compare it to the theory.

## Physical interpretation & saturation mechanisms

Let us move backward a bit, and rerun the first simulation with `k=2` ($k = 2\omega_{pe}/c$).

**2.f** – Plot the species current densities $J_z$ for both electron species, together with the $B_y$ magnetic field component. Also add the $x - v_z$ phase-space diagnostics. Let them evolve with time but keep in the linear phase. Can you give a simple explanation (e.g. making a drawing) of the physical process behind this instability?

**2.g** – Now, go at the saturation stage of the instability. Can you figure out why the instability saturates?

Let us finally investigate the case where the seeded wavelength is larger by taking a smaller wavenumber `k=0.31` ($k = 0.31\omega_{pe}/c$).

**2.h** – Check that the linear phase behaves similarly to the previous case.

**2.i** – What about the saturation phase? Can you figure out what, in that case (longer wavelength), is responsible for the saturation of the instability?

# Project 3

## Simulation set-up

In this project, we will investigate the temporal evolution of a (small amplitude) Langmuir wave propagating into a plasma (again with immobile ions) with density $n_0$ and temperature $T_0 = 10^{-4} m_e c^2$. The wave is initialized as a density and velocity perturbation.

The Languir wave is initialized by prescribing a perturbation on the electron density and (mean) velocity density profiles. The wavevector of the seeded Langmuir wave is defined by the parameter `klde` which denotes $k\,\lambda_{De}$ with $\lambda_{De} = \sqrt{\epsilon_0 T_0/(e^2 n_0)}$. You can check this in the input file `project3.py`.

**3.a** – What is the formula for the Debye length in normalized units (remember we normalize times to the inverse of the electron plasma frequency $\omega_{pe}^{-1}$)?

**3.b** – Do you know why we have introduced the velocity perturbation as $v_x(t=0,x) = \omega n_e(t=0,x)/k$?

## First run

Run the first simulation using `klde=0.1`, i.e. using $k\lambda_{de} = 0.1$. Plot the electrostatic field $E_x$ and electron charge density $\rho$ and hit play to see the time evolution of all quantities.

**3.c** – What do you observe? Can you extract the phase-velocity of the Langmuir wave? How does it compare to the thermal velocity $v_{th} = \sqrt{T_0/m_e} = 0.01\,c$? How does it compare with the theory?

**3.d** – What in the input file allows to define a wave propagating in the forward (positive $x$) direction? What would have happened if one had taken $v_x(t=0,x) = 0$? How could one have injected a wave propagating in the other direction?
**3.e** – Have a look at the energy in the electrostatic field $E_x$. How does it evolve? Can you extract the wave frequency? How does it compare with the theory?

## Additional runs varying $k\lambda_{De}$

Now, we will successfully run the same simulation slowly increasing $k\lambda_{De}$. For the next runs, use `klde=0.2`, `0.3` and `0.4`.

**3.f** – For each of these simulations, plot the energy in the electrostatic $E_x$ field, the electrostatic field $E_x$, charge density $\rho$, velocity distribution ($v_x$ phase space) and $x - v_x$ phase-space distribution.

**3.g** – For each case, extract the wave frequency and phase-velocity. Compare the wave velocity with the thermal one. Also extract, if observed, the damping rate of the wave. Compare your numerical results to the theoretical predictions.

## Physical interpretation

**3.h** – Now, from the *supplemental material* and observing into more details the phase-space distribution, can you infer what is responsible for the damping of the Langmuir wave?

# Project 4

## Simulation set-up

In this last project, the simulation is initialized with a background plasma with density $n_0$ (immobile ions), mean-velocity $\mathbf{v}_0 \sim 0$ and temperature $T_0 = 4 \times 10^{-4} m_e c^2$. In addition to this background plasma, we have initialized an electron beam with density $n_b = 0.05 n_0 \ll n_0$, mean-velocity along the $x$-direction $v_b = 0.12\ c$ and temperature $T_b = T_0$.

**4.a** – Have a look at the input file `project4.py`. How is defined the background (electron) plasma mean-velocity? Why did we not simply take $\mathbf{v}_0 = 0$?
**4.b** – Now look at the species blocks `Species()`. How many species blocks do you find? What happened to the ions? What do you think will happen?

## First run

**4.c** – Run this new simulation. Then, look first at the $E_x$ and $\rho_{\rm plasma}$ and $\rho_{\rm beam}$ at time $t = 0$. Is it what you expect after having answered to question 4.b? Any clue about what happened?

**4.d** – Now, look at the initial (time $t = 0$) $v_x$-distribution. Do you see where the *bump-on-tail* instability name comes from?

**4.e** – Look at the time evolution of the various quantities. What's happening? In particular, extract from your simulation the growth rate of the instability and phase velocity of any excited wave. What does this phase velocity corresponds to?

**4.f** – Did we seed any particular mode to start this instability? From what fluctuations does the instability start?

## Physical interpretation

**4.g** – Can you give a simple interpretation for the linear stage of the instability?

**4.h** – What about the non linear stage? What is, according to you, responsible for the saturation of the instability?

**4.i** – Decrease the beam velocity, e.g. taking $v_b = 0.8c$. What do you observe? Can you explain why?

# Supplemental Material

This supplemental material provides some theoretical support to the practicals. It relies purely on the linear theory of either wave or instabilities. The nonlinear, saturation stage will not be addressed here even though it will be discussed in the practicals.

**Nota bene:** Below are given mainly technical points. The physical interpretation of the different processes studied will be investigated in the practical, or can be found in most text book on plasma physics.

## A. General framework

For the sake of generality, all electromagnetic fields will be described by Maxwell's equations, and Maxwell-Ampère and Maxwell Faraday equations in particular:

$$\frac{1}{c^2}\partial_t \mathbf{E} = -\mu_0 \mathbf{J} + \nabla \times \mathbf{B}\,, \tag{1a}$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}\,, \tag{1b}$$

with $\mathbf{J} = \sum_s \mathbf{J}_s$ the total current density provided by the various species $s$ in the plasma.

Throughout this *supplemental material*, we will focus on linear waves and/or the linear phase of instabilities in the absence of any external electric or magnetic fields $\left[\mathbf{E}^{(0)} = \mathbf{B}^{(0)} = 0\right]$. Furthermore, looking at first order quantities in the form

$$\phi^{(1)} = \tilde{\phi}^{(1)}\, \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$$

leads for the perturbed electric and magnetic fields:

$$\omega\, \mathbf{E}^{(1)} = -\frac{i}{\epsilon_0}\, \mathbf{J}^{(1)} - c^2\, \mathbf{k} \times \mathbf{B}^{(1)}\,, \tag{2a}$$

$$\omega\, \mathbf{B}^{(1)} = \mathbf{k} \times \mathbf{E}^{(1)}\,. \tag{2b}$$

From Eqs. (2a) and (2b), we obtain:

$$\omega^2 \mathbf{E}^{(1)} + c^2\, \mathbf{k} \times \mathbf{k} \times \mathbf{E}^{(1)} = -i\, \frac{\omega}{\epsilon_0}\, \mathbf{J}^{(1)}\,, \tag{3}$$

and making use of the vector identity $\mathbf{A} \times \mathbf{B} \times \mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$:

$$\left[(\omega^2 - c^2\, \mathbf{k}^2)\, \mathbb{I} + c^2\, \mathbf{k} \otimes \mathbf{k}\right] \mathbf{E}^{(1)} = -i\, \frac{\omega}{\epsilon_0}\, \mathbf{J}^{(1)}\,, \tag{4}$$

with $\mathbb{I}$ the identity matrix and $\mathbf{u} \otimes \mathbf{v}$ the dyadic product of vector $\mathbf{u}$ and $\mathbf{v}$ that returns the matrix $\underline{\mathbf{M}}$ with elements $M_{ij} = u_i v_j$.

Now, the trick is to express all (first order) current densities $\mathbf{J}_s^{(1)}$ as a function of $\mathbf{E}^{(1)}$. In a linear theory, one will always be able to rewrite the current density in the form:

$$\mathbf{J}^{(1)} = \underline{\boldsymbol{\sigma}}\, \mathbf{E}^{(1)}$$

where $\underline{\boldsymbol{\sigma}}$ is the conductivity tensor, leading to:

$$\left[\omega^2 \underline{\boldsymbol{\epsilon}} + c^2 \,\left(\mathbf{k} \otimes \mathbf{k} - k^2\, \mathbb{I}\right)\right] \mathbf{E}^{(1)} = 0\,, \tag{5}$$

where we have introduced the permittivity tensor:

$$\underline{\boldsymbol{\epsilon}} = \mathbb{I} + \frac{i}{\epsilon_0 \omega} \underline{\boldsymbol{\sigma}}\,. \tag{6}$$

Without loss of generality, we now consider that the wavevector[1] $\mathbf{k} = (k_\parallel, k_\perp, 0)$, and Eq. (5) can be rewritten in the matrix form:

$$\underline{\mathbf{P}}\, \mathbf{E}^{(1)} = 0\,, \tag{7}$$

with:

$$\underline{\mathbf{P}} = \begin{bmatrix} \omega^2\epsilon_{11} - c^2 k_\perp^2 & \omega^2\epsilon_{12} - c^2 k_\parallel k_\perp & \omega^2\epsilon_{13} \\ \omega^2\epsilon_{21} - c^2 k_\parallel k_\perp & \omega^2\epsilon_{22} - c^2 k_\parallel^2 & \omega^2\epsilon_{23} \\ \omega^2\epsilon_{31} & \omega^2\epsilon_{32} & \omega^2\epsilon_{33} - c^2 k^2 \end{bmatrix} \tag{8}$$

To solve Eq. (7), one needs to cancel the determinant of the *propagator* $\underline{\mathbf{P}}$. Doing so will provide us with the *dispersion relations* for the various waves and/or instabilities at play.

Now, the permittivity tensor needs to be computed for each species so that the total permittivity tensor can be obtained by summing the contribution of each species. This tensor contains all the information of the plasma species. In what follows we will do it first in the framework of a relativistic cold fluid model, then in the framework of the (kinetic) Vlasov equation.

## B. Weibel & two-stream instabilities driven two counter-streaming cold electron plasmas

In this Section, we focus on two instabilities that develop in the presence of counter-streaming flows. For the sake of simplicity, we here consider the simple case of two counter-streaming electron plasmas, each with density $n_0/2$ and opposite velocities $\pm\mathbf{v}_0$, and a neutralizing background of immobile ions with density $n_0$. We will consider the first spatial dimension as that given by $\mathbf{v}_0$, i.e. $\mathbf{v}_0 = (v_0, 0, 0)$.

Both electron species (labeled $s = \pm$) are treated in the framework of a relativistic cold fluid model:

$$\partial_t n_s + \nabla \cdot (n_s \mathbf{v}_s) = 0\,, \tag{9a}$$

$$\partial_t \mathbf{p}_s + (\mathbf{v}_s \cdot \nabla)\mathbf{p}_s = q_s\, \left(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}\right)\,. \tag{9b}$$

We linearise the fluid Eqs. (9a)-(9b) considering all initial (zero-order) quantities homogeneous (and no external fields, as previously assumed). Looking for first order quantities in the form $\phi^{(1)} = \tilde{\phi}^{(1)} \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right]$, leads to:

$$n_s^{(1)} = n_s^{(0)} \frac{\mathbf{k}\cdot\mathbf{v}_s^{(1)}}{\omega_{ds}}\,, \tag{10a}$$

$$\mathbf{p}_s^{(1)} = i\,\frac{q_s}{\omega_{ds}}\,\left(\mathbf{E}^{(1)} + \mathbf{v}_s^{(0)} \times \mathbf{B}^{(1)}\right)\,, \tag{10b}$$

[1]Later on, we will allow the plasma species to flow in the direction $(\pm 1, 0, 0)$.

where we have introduced:

$$\omega_{ds} = \omega - \mathbf{v}_s^{(0)} \cdot \mathbf{k}\,.$$

As previously stated, we wish to derive the equation for the species (first order) density current:

$$\mathbf{J_s}^{(1)} = q_s\, n_s^{(0)}\, \mathbf{v}_s^{(1)} + q_s\, n_s^{(1)}\, \mathbf{v}_s^{(0)}\,. \tag{11}$$

Using Eq. (10a), we get:

$$\mathbf{J_s}^{(1)} = q_s\, n_s^{(0)} \left[\mathbb{I} + \frac{\mathbf{v}_s^{(0)} \otimes \mathbf{k}}{\omega_{ds}}\right] \mathbf{v}_s^{(1)} = q_s n_s^{(0)} \begin{bmatrix} 1 + \frac{v_s^{(0)} k_\parallel}{\omega_{ds}} & \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v}_s^{(1)}\,. \tag{12}$$

To get $\mathbf{v}_s^{(1)}$ we Taylor expand $\mathbf{v}_s$ expressed as a function of the species momentum $\mathbf{p}_s$ around $\mathbf{v}_s^{(0)}$, which leads to:

$$\mathbf{v}_s^{(1)} = \frac{\mathbf{p}_s^{(1)}}{m_s \gamma_{s0}} - \frac{\mathbf{p}_s^{(0)}}{m_s \gamma_{s0}^3} \frac{\mathbf{p}_s^{(0)} \cdot \mathbf{p}_s^{(1)}}{m_s^2 c^2} = \frac{1}{m_s \gamma_{s0}} \left[\mathbb{I} - \mathbf{p}_s^{(0)} \otimes \mathbf{p}_s^{(0)}\right] \mathbf{p}_s^{(1)}\,, \tag{13}$$

with $\gamma_{s0} = \sqrt{1 + \mathbf{p}_s^{(0)2}/(m_s c)^2}$. We rewrite in the matrix form:

$$\mathbf{v}_s^{(1)} = \frac{1}{m_s \gamma_{s0}} \begin{bmatrix} \gamma_{s0}^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_s^{(1)}\,. \tag{14}$$

One then computes $\mathbf{p}_s^{(1)}$ from Eq. (10b) (using Faraday's equation to rewrite the magnetic field in terms of the electric field) as:

$$\mathbf{p}_s^{(1)} = i\, \frac{q_s}{\omega} \left[\mathbb{I} + \frac{\mathbf{k} \otimes \mathbf{v}_s^{(0)}}{\omega_{ds}}\right] \mathbf{E}^{(1)} = i\, \frac{q_s}{\omega} \begin{bmatrix} 1 + \frac{v_s^{(0)} k_\parallel}{\omega_{ds}} & 0 & 0 \\ \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)}\,. \tag{15}$$

Combining Eqs. (12), (14) and (15), one finally gets for the current density:

$$\mathbf{J}_s^{(1)} = i\, \epsilon_0 \frac{\omega_{ps,0}^2}{\gamma_{s0}\omega} \begin{bmatrix} \frac{1}{\gamma_{s0}^2}\left(1 + \frac{v_s^{(0)} k_\parallel}{\omega_{ds}}\right)^2 + \frac{v_s^{(0)2} k_\perp^2}{\omega_{ds}^2} & \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 0 \\ \frac{v_s^{(0)} k_\perp}{\omega_{ds}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)}\,, \tag{16}$$

where we have introduced the species plasma frequency

$$\omega_{ps,0} = \sqrt{\frac{q_s^2 n_s^{(0)}}{\epsilon_0 m_s}}\,. \tag{17}$$

### B.1 Purely transverse instability: the Weibel or filamentation instability

Let us first consider the case where the wavevector $\mathbf{k} = (0, k_\perp = k, 0)$ is perpendicular to the electron flow velocity, so that $\forall s,\ \mathbf{v}_s^{(0)} \cdot \mathbf{k} = 0$ and $\omega_{ds} = \omega$. The total current density (at first order) is obtained by summing over the two ($s = \pm$) species which, we recall, have opposite drift-velocity $\mathbf{v}_\pm^{(0)} = \pm \mathbf{v}_0$. From Eq. (16), one gets[2]:

$$\mathbf{J}^{(1)} = i\,\epsilon_0 \frac{\omega_{p0}^2}{\gamma_0 \omega} \begin{bmatrix} \frac{1}{\gamma_0^2} + \frac{v_0^2 k^2}{\omega^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)}\,. \tag{18}$$

This corresponds to the (total) permittivity:

$$\underline{\boldsymbol{\epsilon}} = \begin{bmatrix} 1 - \frac{\omega_{p0}^2}{\gamma_0\omega^2}\left(\frac{1}{\gamma_0^2} + \frac{v_0^2 k^2}{\omega^2}\right) & 0 & 0 \\ 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0\omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0\omega^2} \end{bmatrix} \tag{19}$$

Injecting Eq. (19) in Eq. (8) leads:

$$\begin{vmatrix} \omega^2 - c^2k^2 - \frac{\omega_{p0}^2}{\gamma_0}\left(\frac{1}{\gamma_0^2} + \frac{v_0^2 k^2}{\omega^2}\right) & 0 & 0 \\ 0 & \omega^2 - \frac{\omega_{p0}^2}{\gamma_0} & 0 \\ 0 & 0 & \omega^2 - c^2k^2 - \frac{\omega_{p0}^2}{\gamma_0} \end{vmatrix} = 0\,. \tag{20}$$

Now, the propagator is a diagonal matrix so that canceling its determinant is straightforward. Two solutions correspond to the standard electrostatic and electromagnetic waves in a plasma and are associated to the electric fields. Of particular importance is the mode corresponding to the electric field aligned with the flow velocity, for which:

$$\omega^2 - \frac{\omega_{p0}^2}{\gamma_0}\left(\frac{1}{\gamma_0^2} + \frac{v_0^2 k^2}{\omega^2}\right) - c^2k^2 = 0\,. \tag{21}$$

Indeed, this equation, which is the *dispersion relation for the Weibel or filamentation instability*, allows for purely imaginary solutions $\omega(k) = i\Gamma_\perp(k)$ (with $\Gamma_\perp(k) > 0$) so that the (flow aligned) electric field grows exponentially at a rate:

$$\Gamma_\perp(k) = \frac{1}{\sqrt{2}} \left[\sqrt{\left(c^2k^2 + \frac{\omega_{p0}^2}{\gamma_0^3}\right)^2 + 4\frac{\omega_{pe}^2}{\gamma_0}\,v_0^2k^2} - \left(c^2k^2 + \frac{\omega_{p0}^2}{\gamma_0^3}\right)\right]^{1/2}\,. \tag{22}$$

Note that this growth rate corresponds to the electric field component aligned with the flow velocity. Hence (see e.g. Faraday's equation), the growth of this electric field will be accompanied by a growth of a magnetic field that is transverse to both, the wavevector $\mathbf{k}$ and $\mathbf{v}_0$. This instability is thus of an electromagnetic nature.

**Exercise** – Show that in the small $k$ limit, the growth rate of the instability increases linearly with $k$ as $\Gamma(k \ll 1) \rightarrow \gamma_0\, v_0\, k$. Then, in the large $k$ limit, show that the growth rate asymptotically reaches the maximum value $\Gamma(k \gg 1) = v_0\omega_{p0}/(c\sqrt{\gamma_0})$.

[2] Note that $\omega_{p0} = \sqrt{2}\,\omega_{p\pm,0}$ corresponds to the overall plasma frequency computed at $n_0$.

### B.2 Purely longitudinal instability: the two-stream instability

Let us now consider the case where the wavevector $\mathbf{k} = (k_\parallel = k, 0, 0)$ is aligned with the electron flow velocity, so that $\forall s,\ \mathbf{v}_s^{(0)} \cdot \mathbf{k} = \pm v_0 k$ and $\omega_{d\pm} = \omega \pm v_0 k$. The total current density (at first order) is obtained by summing over the two ($s = \pm$) species, leading to:

$$\mathbf{J}^{(1)} = i\epsilon_0 \frac{\omega_{p0}^2}{\gamma_0 \omega} \begin{bmatrix} \frac{1}{2\gamma_0^2}\left[\frac{\omega^2}{(\omega - v_0 k)^2} + \frac{\omega^2}{(\omega + v_0 k)^2}\right] & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{E}^{(1)} \tag{23}$$

and the (total) permittivity:

$$\underline{\boldsymbol{\epsilon}} = \begin{bmatrix} 1 - \frac{\omega_{p0}^2}{2\gamma_0^3}\left[\frac{1}{(\omega - v_0 k)^2} + \frac{1}{(\omega + v_0 k)^2}\right] & 0 & 0 \\ 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0 \omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{p0}^2}{\gamma_0 \omega^2} \end{bmatrix} . \tag{24}$$

Injecting Eq. (24) in Eq. (8) leads to:

$$\begin{vmatrix} \omega^2 - \frac{\omega_{p0}^2}{2\gamma_0^3}\left[\frac{\omega^2}{(\omega - v_0 k)^2} + \frac{\omega^2}{(\omega + v_0 k)^2}\right] & 0 & 0 \\ 0 & \omega^2 - c^2 k^2 - \frac{\omega_{p0}^2}{\gamma_0} & 0 \\ 0 & 0 & \omega^2 - c^2 k^2 - \frac{\omega_{p0}^2}{\gamma_0} \end{vmatrix} = 0 . \tag{25}$$

Here again, we find some trivial (propagating wave) solutions for the $y$ and $z$ components of the electric field. In addition, cancelling the first (1,1) term in Eq. (25) provides the *dispersion relation for the two-stream instability*:

$$\omega^4 - \left(2v_0^2 k^2 + \frac{\omega_{p0^2}}{\gamma_0^3}\right)\omega^2 + \left(v_0^2 k^2 - \frac{\omega_{p0}^2}{\gamma_0^3}\right) v_0^2 k^2 = 0 . \tag{26}$$

Note that this instability involves the electric field parallel to the initial flow velocity only. It is accompanied by a charge density perturbation (through e.g. Poisson equation) but is not associated to any magnetic field perturbation. The two-stream instability is therefore of a purely electrostatic nature.

Solving for $\omega^2$ gives:

$$\omega^2 = \frac{1}{2}\left[\frac{\omega_{p0}^2}{\gamma_0^3} + 2v_0^2 k^2 \pm \frac{\omega_{p0}}{\gamma_0^{3/2}}\sqrt{\frac{\omega_{p0}^2}{\gamma_0^3} + 8v_0^2 k^2}\right] . \tag{27}$$

It turns out that, whenever $l_0 k < 1$, with $l_0 = \gamma_0^{3/2} v_0/\omega_{p0}$, $\omega = i\,\Gamma_\parallel(k)$, where $\Gamma_\parallel(k) > 0$ is the growth rate for the two-stream instability.

**Exercise** – (i) Show that the maximum growth rate for the two-stream instability is maximum for $l_0 k = \sqrt{6}/4$, and is then $\Gamma_{\parallel,\max} = \omega_{p0}/(2/\sqrt{2}\gamma_0^{3/2})$. (ii) Compare the growth rate of the two-stream and Weibel/filamentation instabilities at small ($\gamma_0 \sim 1$) and large ($\gamma_0 \ll 1$). Which instability do you expect to dominate in the presence of relativistic flows?

## C. An exemple of kinetic waves & instability: the Landau damping of Langmuir waves and the bump-on-tail instability

Let us now investigate the case of kinetic waves and instabilities. To do so, the plasma species are modelled using the Vlasov equation (for simplicity we restrict ourselves to the non-relativistic limit):

$$\partial_t f_s + \mathbf{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f_s = 0\,. \tag{28}$$

In the absence of any external electromagnetic fields, and considering all perturbed quantities $\phi^{(1)} = \tilde{\phi}^{(1)}\,\exp\left[i(\mathbf{k}\cdot\mathbf{x}-\omega t)\right]$, one gets at first order:

$$f_s^{(1)} = -i\frac{q_s}{m_s}\left(\mathbf{E}^{(1)}+\mathbf{v}\times\mathbf{B}^{(1)}\right)\cdot\frac{\nabla_v f_s^{(0)}}{\omega-\mathbf{k}\cdot\mathbf{v}}\,, \tag{29}$$

where $f_s^{(0)}$ is the initial distribution function which we will assume to be a Maxwellian with temperature $T_s$ and mean velocity $\mathbf{v}_s^{(0)}$:

$$f_s^{(0)}(\mathbf{v}) = n_s^{(0)}\left(\frac{m_s}{2\pi T_s}\right)^{\frac{3}{2}}\exp\left(-\frac{m_s\,(\mathbf{v}-\mathbf{v}_s^{(0)})^2}{2T_s}\right) \tag{30}$$

In what follows, we will further restrict our analysis to (i) purely electrostatic perturbations, $\mathbf{B}^{(1)}=0$ in Eq. (29), and (ii) species drift (mean) velocities $\mathbf{v}_s^{(0)}$ aligned with the wavevector $\mathbf{k} = (k,0,0)$. It follows that only the (first order) electric field and current densities aligned with $\mathbf{k}$, $E_\parallel^{(1)}$ and $J_{s,\parallel}^{(1)}$, are of importance for the problem at hand. One thus focuses on expressing the current density perturbation in the form:

$$J_{s,\parallel}^{(1)} = q_s\int d^3v\,v_1 f_s^{(1)} = -i\frac{q_s^2}{m_s}\int d^3v\frac{v_1\,\partial f_s^{(0)}/\partial v_1}{\omega-kv_1}\,E_\parallel^{(1)}\,. \tag{31}$$

Introducing the reduced distribution function:

$$\bar{f}_s^{(0)} = \frac{1}{n_s^{(0)}}\int dv_2 dv_3 f_s^{(0)} = \sqrt{\frac{m_s}{2\pi T_s}}\,\exp\left(-\frac{m_s\,(v_1-v_s^{(0)})^2}{2T_s}\right)\,, \tag{32}$$

one finally gets for the current density:

$$J_{s,\parallel}^{(1)} = -i\frac{q_s^2 n_s^{(0)}}{m_s}\int dv_1\frac{v_1\,\partial\bar{f}_s^{(0)}/\partial v_1}{\omega-kv_1}\,E_\parallel^{(1)} = -i\epsilon_0\omega\frac{\omega_{ps,0}^2}{k^2}\int dv_1\frac{\partial\bar{f}_s^{(0)}/\partial v_1}{\omega/k-v_1}\,E_\parallel^{(1)}\,. \tag{33}$$

One then gets the permittivity:

$$\epsilon_s^{\parallel} = 1 + \frac{\omega_{ps,0}^2}{k^2}\int dv_1\frac{\partial\bar{f}_s^{(0)}/\partial v_1}{\omega/k-v_1}\,. \tag{34}$$

### C.1 Landau damping of Langmuir waves

Langmuir waves are electrostatic perturbations propagating in a plasma. Let us thus consider a plasma with (immobile ions and) an homogeneous density $n_0$, zero mean velocity, and temperature $T_0$. The plasma permittivity then reads:

$$\epsilon^{\parallel}(\omega,k) = 1 + \frac{\omega_{p0}^2}{k^2}\int dv_1\frac{\partial\bar{f}_s^{(0)}/\partial v_1}{\omega/k-v_1}\,. \tag{35}$$

The real part of the permittivity provides us with the dispersion relation for the Langmuir waves. It can be easily computed in the limit $kv_{th}/\omega \ll 1$, and after some algebra, leads to the (approximated) *dispersion relation for the Langmuir waves*:

$$\omega(k) \simeq \omega_{p0}\left(1+\frac{3}{2}k^2\lambda_{De}^2\right). \tag{36}$$

**Exercise** – Compute the phase and group velocity for the Langmuir waves? What happens in the limit $T_0 \to 0$?

In addition, the presence of a singularity in the integral in Eq. (35) leads to an imaginary contribution to the permittivity:

$$\mathrm{Im}\,\epsilon^{\|}(\omega,k) = -\pi\,\frac{\omega_{p0}}{k^2}\int dv_1 \partial_{v1}\bar{f}^{(0)}\,\delta(\omega/k - v_1) = -\pi\,\frac{\omega_{p0}}{k^2}\,\frac{\partial \bar{f}^{(0)}}{\partial v_1}\bigg|_{v_1=\omega/k}. \tag{37}$$

which gives rise to the damping of the Langmuir wave at a rate:

$$\gamma_L(k) = \frac{\mathrm{Im}\,\epsilon^{\|}}{\partial_\omega \mathrm{Re}\,\epsilon^{\|}} = \sqrt{\frac{8}{\pi}}\,\frac{\omega_{p0}}{k^3\lambda_{De}^3}\,\exp\left(-\frac{3}{2}-\frac{1}{2k^2\lambda_{De}^2}\right). \tag{38}$$

### C.2 Bump on Tail instability

The bump on tail instability refers to the (electrostatic) instability that develops in the presence of (i) a background plasma with zero (or close to zero) mean velocity and temperature $T_0$, and of (ii) a beam, with density $n_b \ll n_0$, non zero velocity $v_b$ (in the direction parallel to the instability wavevector) and temperature $T_b$.

The presence of the electron beam will drive a Langmuir wave supported by the background plasma. Its dispersion relation is thus given by:

$$\omega^2 \simeq \omega_{p0}^2\left(1+3k^2\lambda_{De}^2\right). \tag{39}$$

This wave will be damped by Landau damping (trapping of the background plasma electron), but at the same time can pick up energy from the beam electrons, which gives rise to the *bump in tail instability*. To develop, the growth rate of the instability $\propto \partial_{v_1} f_b^{(0)}|_{v_1=\omega/k} > 0$ needs to be larger than the rate of Landau damping $\propto \partial_{v_1} f_0^{(0)}|_{v_1=\omega/k} < 0$. This leads to a threshold for the instability that cas be casted in the form:

$$\frac{n_b}{n_0}\left(\frac{v_0}{v_{th,b}}\right)^2 > \left(\frac{v_0}{v_{th,0}}\right)^3 \exp\left(-\frac{mv_0^2}{2T_0}\right). \tag{40}$$

**Exercise** – Following Sec. C.1, derive in the limit $\mathrm{Im}\,\omega \ll \mathrm{Re}\,\omega$ the dispersion relation for this instability.

# References

[1] J. Derouillat, A. Beck, F. Pérez, T. Vinci, M. Chiaramello, A. Grassi, M. Flé, G. Bouchard, I. Plotnikov, N. Aunai, J. Dargent, C. Riconda, and M. Grech. Smilei : A collaborative, open-source, multi-purpose particle-in-cell code for plasma simulation. *Computer Physics Communications*, 222:351 – 373, 2018.