The following document, taken from

P.M.Bellan, Fundamentals of plasma physics, Cambridge University Press, 2006.

derives the Landau damping of electron plasma waves. Pages 180-192, contains the treatment of the Landau problem similarly to what has been presented during the lecture.

The treatment of Landau damping of electron plasma waves is presented on p. 192-197. There was no time to present this during the lecture.

5.3 The Landau problem

A plasma wave behavior of great philosophical interest and of great practical importance can now be investigated. Before doing so, we recall three seemingly disconnected results obtained thus far, namely:

1. When the exchange of energy between charged particles and a simple one-dimensional electrostatic wave with dependence $\sim \exp(ikx - i\omega t)$ was considered, the particles were categorized into two general classes, trapped and untrapped, and it was found that untrapped particles tended to be dragged towards the wave phase velocity. Thus, untrapped particles moving slower than the wave gain kinetic energy, whereas untrapped particles moving faster lose kinetic energy. This has the consequence that if there are more slow than fast particles, the particles gain net kinetic energy overall and this gain presumably comes at the expense of the wave. Conversely, if there are more fast than slow particles, net energy flows from the particles to the wave.
2. When electrostatic plasma waves in an unmagnetized, uniform, stationary plasma were considered, it was found that wave behavior was characterized by a dispersion relation $1 + \chi_e(\omega, k) + \chi_i(\omega, k) = 0$, where $\chi_\sigma(\omega, k)$ is the susceptibility of each species σ. As sketched in Fig. 4.1 these susceptibilities had simple limiting forms when $\omega/k \ll \sqrt{\kappa T_{\sigma 0}/m_\sigma}$ (isothermal limit) and when $\omega/k \gg \sqrt{\kappa T_{\sigma 0}/m_\sigma}$ (adiabatic limit), but the fluid analysis failed when $\omega/k \sim \sqrt{\kappa T_{\sigma 0}/m_\sigma}$ and the susceptibilities became undefined.
3. When the behavior of interacting beams of particles was considered, it was found that under certain conditions a fast-growing instability would develop.

The analysis of the Landau problem, to be presented in the remainder of this chapter, will show that these three results are both interrelated and part of a larger picture.

5.3.1 *Attempt to solve the linearized Vlasov–Poisson system of equations using Fourier analysis*

The method for manipulating fluid equations to find wave solutions was as follows: (i) the relevant fluid equations were linearized, (ii) a perturbation $\sim \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ was assumed, (iii) the system of partial differential equations was transformed into a system of algebraic equations, and then finally (iv) the roots of the determinant of the system of algebraic equations provided the dispersion relations that characterized the various wave solutions.

It seems reasonable to use this method again in order to investigate waves from the Vlasov point of view. However, it will be seen that this approach *fails* and that, instead, a more complicated Laplace transform technique must be used. However, once the underlying difference between the Laplace and Fourier transform techniques has been identified, it is possible to go back and "patch up" the Fourier technique. Although perhaps not entirely elegant, this patching approach turns out to be a reasonable compromise in that it incorporates both the simplicity of the Fourier method and the correct mathematics/physics of the Laplace method.

The Fourier method will now be presented and, to highlight how this method fails, the simplest relevant example will be considered, namely a one-dimensional, unmagnetized plasma with a stationary Maxwellian equilibrium. The ions are assumed to be so massive as to be immobile and the ion density is assumed to equal the electron equilibrium density. The electrostatic electric field $E = -\partial\phi/\partial x$ is therefore zero in equilibrium because there is charge neutrality in equilibrium. Since ions do not move there is no need to track ion dynamics. Thus, all perturbed quantities refer to electrons and so it is redundant to label these with a subscript "e." In order to have a well-defined, physically meaningful problem, the equilibrium electron velocity distribution is assumed to be Maxwellian, i.e.,

$$f_0(v) = n_0 \frac{1}{\pi^{1/2} v_T} \mathrm{e}^{-v^2/v_T^2}, \tag{5.24}$$

where $v_T \equiv \sqrt{2\kappa T/m}$.

The one-dimensional, unmagnetized Vlasov equation is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{q}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0 \tag{5.25}$$

and linearization of this equation gives

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{q}{m}\frac{\partial \phi_1}{\partial x}\frac{\partial f_0}{\partial v} = 0. \tag{5.26}$$

Because the Vlasov equation describes evolution in phase-space, v is an *independent* variable just like x and t. Assuming a normal mode dependence $\sim \exp(\mathrm{i}kx - \mathrm{i}\omega t)$, Eq. (5.26) becomes

$$-\mathrm{i}(\omega - kv)f_1 - \mathrm{i}k\phi_1 \frac{q}{m}\frac{\partial f_0}{\partial v} = 0, \tag{5.27}$$

which gives

$$f_1 = -\frac{k}{(\omega - kv)}\frac{q}{m}\frac{\partial f_0}{\partial v}\phi_1. \tag{5.28}$$

The electron density perturbation is

$$n_1 = \int_{-\infty}^{\infty} f_1 \mathrm{d}v = -\frac{q}{m}\phi_1 \int_{-\infty}^{\infty} \frac{k}{(\omega - kv)}\frac{\partial f_0}{\partial v}\mathrm{d}v, \tag{5.29}$$

a relationship between n_1 and ϕ_1. Another relationship between n_1 and ϕ_1 is Poisson's equation

$$\frac{\partial^2 \phi_1}{\partial x^2} = -\frac{n_1 q}{\varepsilon_0}. \tag{5.30}$$

Replacing $\partial/\partial x$ by $\mathrm{i}k$, Eq. (5.30) becomes

$$k^2 \phi_1 = \frac{n_1 q}{\varepsilon_0}. \tag{5.31}$$

Combining Eqs. (5.29) and (5.31) gives the dispersion relation

$$1 + \frac{q^2}{k^2 m \varepsilon_0} \int_{-\infty}^{\infty} \frac{k}{(\omega - kv)} \frac{\partial f_0}{\partial v} \mathrm{d}v = 0. \tag{5.32}$$

This can be written more elegantly by substituting for f_0 using Eq. (5.24), defining the non-dimensional particle velocity $\xi = v/v_T$, and the non-dimensional phase velocity $\alpha = \omega/kv_T$ to give

$$1 - \frac{1}{2k^2\lambda_D^2} \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \mathrm{d}\xi \frac{1}{(\xi - \alpha)} \frac{\partial}{\partial \xi} \mathrm{e}^{-\xi^2} = 0 \tag{5.33}$$

or

$$1 + \chi = 0, \tag{5.34}$$

where the electron susceptibility is

$$\chi = -\frac{1}{2k^2\lambda_D^2} \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \mathrm{d}\xi \frac{1}{(\xi - \alpha)} \frac{\partial}{\partial \xi} \mathrm{e}^{-\xi^2}. \tag{5.35}$$

In contrast to the earlier two-fluid wave analysis, where in effect the zeroth, first, and second moments of the Vlasov equation were combined (continuity equation, equation of motion, and equation of state), here only the Vlasov equation is involved. Thus the Vlasov equation contains all the information of the moment equations and more. The Vlasov method therefore seems a simpler and more direct way for calculating the susceptibilities than the fluid method, except for a serious difficulty: the integral in Eq. (5.35) is mathematically ill-defined because the denominator vanishes when $\xi = \alpha$ (i.e., when $\omega = kv_T$). Because it is not clear how to deal with this singularity, the ξ integral cannot be evaluated and the Fourier method fails. This is essentially the same as the problem encountered in fluid analysis when ω/k became comparable to $\sqrt{\kappa T/m}$.

5.3.2 *Landau method: Laplace transforms*

Landau (1946) argued that the Fourier problem presented above is ill-posed and showed that the linearized Vlasov–Poisson problem should be treated as an *initial-value* problem, rather than as a normal mode problem. The initial-value point of view is conceptually related to the analysis of single particle motion in sawtooth or sine waves. Before presenting the Landau analysis of the linearized Vlasov–Poisson problem, certain important features of Laplace transforms will now be reviewed.

The Laplace transform of a function $\psi(t)$ is defined as

$$\tilde{\psi}(p) = \int_0^\infty \psi(t) \mathrm{e}^{-pt} \mathrm{d}t \tag{5.36}$$

and can be considered as a "half of a Fourier transform" since the time integration starts at $t = 0$ rather than $t = -\infty$. Caution is required regarding the convergence of this integral for situations where $\psi(t)$ contains exponentially growing terms.

Suppose such exponentially growing terms exist. As $t \to \infty$, the fastest growing term, say $\exp(\gamma t)$, will dominate all other terms contributing to $\psi(t)$. The integral in Eq. (5.36) will then diverge as $t \to \infty$, unless a *restriction* is imposed on the real part of p. In particular, if it is *required* that Re $p > \gamma$, then the decaying $\exp(-pt)$ factor will always overwhelm the growing $\exp(\gamma t)$ factor so that the integral in Eq. (5.36) will converge. These issues of convergence are ignored in Fourier transforms where it is implicitly assumed that the function being transformed has neither exponentially growing terms (which diverge at $t = \infty$) nor exponentially decaying terms (which diverge at $t = -\infty$).

Thus, the integral transform in Eq. (5.36) is defined *only* for Re $p > \gamma$. To emphasize this restriction, Eq. (5.36) is rewritten as

$$\tilde{\psi}(p) = \int_0^\infty \psi(t) \mathrm{e}^{-pt} \mathrm{d}t, \quad \mathrm{Re}\, p > \gamma, \tag{5.37}$$

where γ is the fastest growing exponential term contained in $\psi(t)$. Since p is typically complex, Eq. (5.37) means that $\tilde{\psi}(p)$ is *only* defined in that part of the complex p-plane lying to the *right* of γ as sketched in Fig. 5.3(a). Whenever $\tilde{\psi}(p)$ is used, one must be very careful to avoid venturing outside the region in p-space where $\tilde{\psi}(p)$ is defined (this restriction will later become an important issue).

To construct an inverse transform, consider the integral

$$g(t) = \int_C \mathrm{d}p\, \tilde{\psi}(p) \mathrm{e}^{pt}. \tag{5.38}$$

This integral is ambiguously defined for now because the integration contour C is unspecified. However, whatever integration contour is ultimately selected must not venture into regions where $\tilde{\psi}(p)$ is undefined. Thus, an allowed integration path must have Re $p > \gamma$. Substitution of Eq. (5.37) into Eq. (5.38) and interchanging the order of integration gives

$$g(t) = \int_0^\infty \mathrm{d}t' \int_C \mathrm{d}p\, \psi(t') \mathrm{e}^{p(t-t')}, \quad \mathrm{Re}\, p > \gamma. \tag{5.39}$$

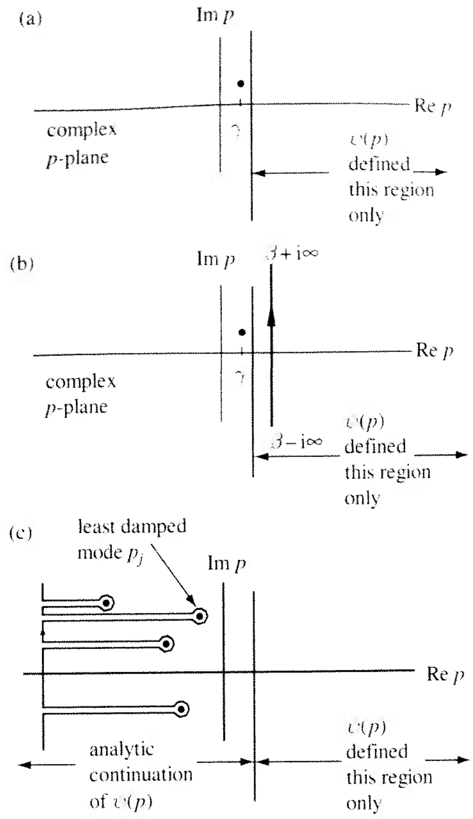

Fig. 5.3 Contours in complex p-plane.

A useful integration path C for the p integral will now be determined. Recall from the theory of Fourier transforms that the Dirac delta function can be expressed as

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{5.40}$$

which is an integral along the real ω axis so that ω is always real. The integration path for Eq. (5.39) will now be chosen such that the real part of p stays constant, say at a value β that is larger than γ, while the imaginary part of p goes from $-\infty$ to ∞. This path is shown in Fig. 5.3(b), and is called the Bromwich contour.

For this choice of path, Eq. (5.39) becomes

$$\begin{aligned} g(t) &= \int_0^{\infty} \mathrm{d}t' \int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty} \mathrm{d}(p_r + \mathrm{i}p_i)\, \psi(t')\mathrm{e}^{(p_r+\mathrm{i}p_i)(t-t')} \\ &= \mathrm{i}\int_0^{\infty} \mathrm{d}t' \mathrm{e}^{\beta(t-t')}\psi(t') \int_{-\infty}^{\infty} \mathrm{d}p_i\, \mathrm{e}^{\mathrm{i}p_i(t-t')} \end{aligned}$$

$$= 2\pi\mathrm{i}\int_0^\infty \mathrm{d}t' \mathrm{e}^{\beta(t-t')}\psi(t')\delta(t-t')$$
$$= 2\pi\mathrm{i}\psi(t), \tag{5.41}$$

where Eq. (5.40) has been used. Thus, $\psi(t) = (2\pi\mathrm{i})^{-1}g(t)$ and so the inverse of the Laplace transform is

$$\psi(t) = \frac{1}{2\pi\mathrm{i}}\int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty} \mathrm{d}p\,\psi(p)\mathrm{e}^{pt}, \quad \beta > \gamma. \tag{5.42}$$

Before returning to physics, recall another peculiarity of Laplace transforms, namely the transformation procedure for derivatives. The Laplace transform of $\mathrm{d}\psi/\mathrm{d}t$ may be simplified by integrating by parts to give

$$\int_0^\infty \mathrm{d}t\,\frac{\mathrm{d}\psi}{\mathrm{d}t}\mathrm{e}^{-pt} = \left[\psi(t)\mathrm{e}^{-pt}\right]_0^\infty + p\int_0^\infty \mathrm{d}t\,\psi(t)\mathrm{e}^{-pt} = p\tilde{\psi}(p) - \psi(0). \tag{5.43}$$

Unlike Fourier transforms, here the *initial value* forms part of the transform. Thus, Laplace transforms contain information about the initial value and so should be better suited than Fourier transforms for investigating initial value problems. The importance of the initial value was also evident in the Chapter 3 analysis of particle motion in sawtooth or sine wave potentials.

The requisite mathematical tools are now in hand for investigating the Vlasov–Poisson system and its dependence on initial value. To obtain extra insights with little additional effort, the analysis is extended to the more general situation of a three-dimensional plasma where ions are allowed to move. Again, electrostatic waves are considered, and it is assumed that the equilibrium plasma is stationary, spatially uniform, neutral, and unmagnetized.

The equilibrium velocity distribution of each species is assumed to be a three-dimensional Maxwellian distribution function

$$f_{\sigma 0}(\mathbf{v}) = n_{\sigma 0}\left(\frac{m_\sigma}{2\pi\kappa T_\sigma}\right)^{3/2}\exp\left(-m_\sigma v^2/2\kappa T_\sigma\right). \tag{5.44}$$

The equilibrium electric field is assumed to be zero so that the equilibrium potential is a constant chosen to be zero. It is further assumed that at $t = 0$ there exists a small perturbation of the distribution function and that this perturbation evolves in time so that at later times

$$f_\sigma(\mathbf{x},\mathbf{v},t) = f_{\sigma 0}(\mathbf{v}) + f_{\sigma 1}(\mathbf{x},\mathbf{v},t). \tag{5.45}$$

The linearized Vlasov equation for each species is therefore

$$\frac{\partial f_{\sigma 1}}{\partial t} + \mathbf{v}\cdot\nabla f_{\sigma 1} - \frac{q_\sigma}{m_\sigma}\nabla\phi_1\cdot\frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} = 0. \tag{5.46}$$

All perturbed quantities are assumed to have the spatial dependence $\sim \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})$: this is equivalent to Fourier transforming in space. Equation (5.46) becomes

$$\frac{\partial f_{\sigma 1}}{\partial t} + \mathrm{i}\mathbf{k}\cdot\mathbf{v} f_{\sigma 1} - \frac{q_\sigma}{m_\sigma}\phi_1 \mathrm{i}\mathbf{k}\cdot\frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} = 0. \tag{5.47}$$

Laplace transforming in time gives

$$(p+\mathrm{i}\mathbf{k}\cdot\mathbf{v})\tilde{f}_{\sigma 1}(\mathbf{v},p) - f_{\sigma 1}(\mathbf{v},0) - \frac{q_\sigma}{m_\sigma}\tilde{\phi}_1(p)\mathrm{i}\mathbf{k}\cdot\frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} = 0, \tag{5.48}$$

which may be solved for $\tilde{f}_{\sigma 1}(\mathbf{v},p)$ to give

$$\tilde{f}_{\sigma 1}(\mathbf{v},p) = \frac{1}{(p+\mathrm{i}\mathbf{k}\cdot\mathbf{v})}\left[f_{\sigma 1}(\mathbf{v},0) + \frac{q_\sigma}{m_\sigma}\tilde{\phi}_1(p)\mathrm{i}\mathbf{k}\cdot\frac{\partial f_{\sigma 0}}{\partial \mathbf{v}}\right]. \tag{5.49}$$

This is similar to Eq. (5.28), except that now the Laplace variable p occurs instead of the Fourier variable $-\mathrm{i}\omega$ and also the initial value $f_{\sigma 1}(\mathbf{v},0)$ appears. As before, Poisson's equation can be written as

$$\nabla^2\phi_1 = -\frac{1}{\varepsilon_0}\sum_\sigma q_\sigma n_{\sigma 1} = -\frac{1}{\varepsilon_0}\sum_\sigma q_\sigma \int \mathrm{d}^3v f_{\sigma 1}(\mathbf{x},\mathbf{v},t). \tag{5.50}$$

Replacing $\nabla \to \mathrm{i}\mathbf{k}$ and Laplace transforming with respect to time, Poisson's equation becomes

$$k^2\tilde{\phi}_1(p) = \frac{1}{\varepsilon_0}\sum_\sigma q_\sigma \int \mathrm{d}^3v \tilde{f}_{\sigma 1}(\mathbf{v},p). \tag{5.51}$$

Substitution of Eq. (5.49) into the right-hand side of Eq. (5.51) gives

$$k^2\tilde{\phi}_1(p) = \frac{1}{\varepsilon_0}\sum_\sigma q_\sigma \int \mathrm{d}^3v \left\{\frac{f_{\sigma 1}(\mathbf{v},0) + \dfrac{q_\sigma}{m_\sigma}\tilde{\phi}_1(p)\mathrm{i}\mathbf{k}\cdot\dfrac{\partial f_{\sigma 0}}{\partial \mathbf{v}}}{(p+\mathrm{i}\mathbf{k}\cdot\mathbf{v})}\right\}, \tag{5.52}$$

which is similar to Eq. (5.32) except that $-\mathrm{i}\omega \to p$ and the initial value appears. Equation (5.52) may be solved for $\tilde{\phi}_1(p)$ to give

$$\tilde{\phi}_1(p) = \frac{N(p)}{D(p)}, \tag{5.53}$$

where the numerator is

$$N(p) = \frac{1}{k^2\varepsilon_0}\sum_\sigma q_\sigma \int \mathrm{d}^3v \frac{f_{\sigma 1}(\mathbf{v},0)}{(p+\mathrm{i}\mathbf{k}\cdot\mathbf{v})} \tag{5.54}$$

and the denominator is

$$D(p) = 1 - \frac{1}{k^2}\sum_\sigma \frac{q_\sigma^2}{\varepsilon_0 m_\sigma}\int \mathrm{d}^3v \frac{\mathrm{i}\mathbf{k}\cdot\dfrac{\partial f_{\sigma 0}}{\partial \mathbf{v}}}{(p+\mathrm{i}\mathbf{k}\cdot\mathbf{v})}. \tag{5.55}$$

Note that the denominator is similar to Eq. (5.32). All that has to be done now is take the inverse Laplace transform of Eq. (5.53) to obtain

$$\phi_1(t) = \frac{1}{2\pi \mathrm{i}} \int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty} \mathrm{d}p \, \frac{N(p)}{D(p)} \mathrm{e}^{pt}, \tag{5.56}$$

where β is chosen to be larger than the fastest growing exponential term in $N(p)/D(p)$.

This is an exact formal solution to the problem. However, because of the complexity of $N(p)$ and $D(p)$ it is impossible to evaluate the integral in Eq. (5.56). Nevertheless, it turns out to be feasible to evaluate the long-time asymptotic limit of this integral and, for practical purposes, this is a sufficient answer to the problem.

5.3.3 The relationship between poles, exponential functions, and analytic continuation

Before evaluating Eq. (5.56), it is useful to examine the relationship between exponentially growing/decaying functions, Laplace transforms, poles, residues, and analytic continuation. This relationship is demonstrated by considering the exponential function

$$f(t) = \mathrm{e}^{qt}, \tag{5.57}$$

where q is a complex constant. If the real part of q is positive, then the amplitude of $f(t)$ is exponentially growing, whereas if the real part of q is negative, the amplitude of $f(t)$ is exponentially decaying. Now, calculate the Laplace transform of $f(t)$; it is

$$\tilde{f}(p) = \int_0^\infty \mathrm{e}^{(q-p)t} \mathrm{d}t = \frac{1}{p-q}, \quad \text{defined only for Re } p > \text{Re } q. \tag{5.58}$$

Let us examine the Bromwich contour integral for $\tilde{f}(p)$ and temporarily call this integral $F(t)$; evaluation of $F(t)$ ought to yield $F(t) = f(t)$. Thus, we define

$$F(t) = \frac{1}{2\pi \mathrm{i}} \int_{\beta-\mathrm{i}\infty}^{\beta+\mathrm{i}\infty} \mathrm{d}p \tilde{f}(p) \mathrm{e}^{pt}, \quad \beta > \text{Re } q. \tag{5.59}$$

If the Bromwich contour could be closed in the left-hand p-plane, the integral could easily be evaluated using the method of residues but closure of the contour to the left is forbidden because of the restriction that $\beta > \text{Re } q$. This annoyance may be overcome by constructing a new function $\hat{f}(p)$ that:

1. equals $\tilde{f}(p)$ in the region $\beta > \text{Re } q$,
2. is *also* defined in the region $\beta < \text{Re } q$, and
3. is analytic.

Integration of $\hat{f}(p)$ along the Bromwich contour gives the same result as does integration of $\tilde{f}(p)$ along the same contour because the two functions are *identical* along this contour (cf. stipulation (1) above). Thus, it is seen that

$$F(t) = \frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} dp\hat{f}(p)e^{pt}, \tag{5.60}$$

but now there is no restriction on which part of the p-plane may be used. So long as the end points are kept fixed and no poles are crossed, the path of integration of an analytic function can be arbitrarily deformed. This is because the difference between the original path and a deformed path is a closed contour, which integrates to zero if it does not enclose any poles. Because $\hat{f}(p) \to 0$ at the endpoints $\beta \pm \infty$, the integration path of $\hat{f}(p)$ can be deformed into the left-hand plane as long as $\hat{f}(p)$ remains analytic (i.e., does not jump over any poles or branch cuts). How can this magic function $\hat{f}(p)$ be constructed?

The answer is simple: we *define* a function $\hat{f}(p)$ having the identical functional form as $\tilde{f}(p)$, but *without* the restriction that Re $p >$ Re q. Thus, the analytic continuation of

$$\tilde{f}(p) = \frac{1}{p-q}, \quad \text{defined only for } \mathrm{Re}\, p > \mathrm{Re}\, q, \tag{5.61}$$

is simply

$$\hat{f}(p) = \frac{1}{p-q}, \quad \text{defined for all } p, \text{ provided } \hat{f}(p) \text{ remains analytic.} \tag{5.62}$$

The Bromwich contour can now be deformed into the left-hand plane as shown in Fig. 5.4. Because $\exp(pt) \to 0$ for positive t and negative Re p, the integration contour can be closed by an arc that goes to the left (cf. Fig. 5.4) into the region where Re $p \to -\infty$. The resulting contour encircles the pole at $p = q$ and so the integral can be evaluated using the method of residues as follows:

$$F(t) = \frac{1}{2\pi i}\oint \frac{1}{p-q}e^{pt}dp = \lim_{p\to q} 2\pi i(p-q)\left[\frac{1}{2\pi i(p-q)}e^{pt}\right] = e^{qt}. \tag{5.63}$$

This simple example shows that while the Bromwich contour formally gives the inverse Laplace transform of $\tilde{f}(p)$, the Bromwich contour by itself does not allow use of the method of residues, since the poles of interest are located in the left-hand complex p-plane where $\tilde{f}(p)$ is undefined. However, analytic continuation of $f(p)$ allows deformation of the Bromwich contour into the formerly forbidden area, and then the inverse transform may be easily evaluated using the method of residues.

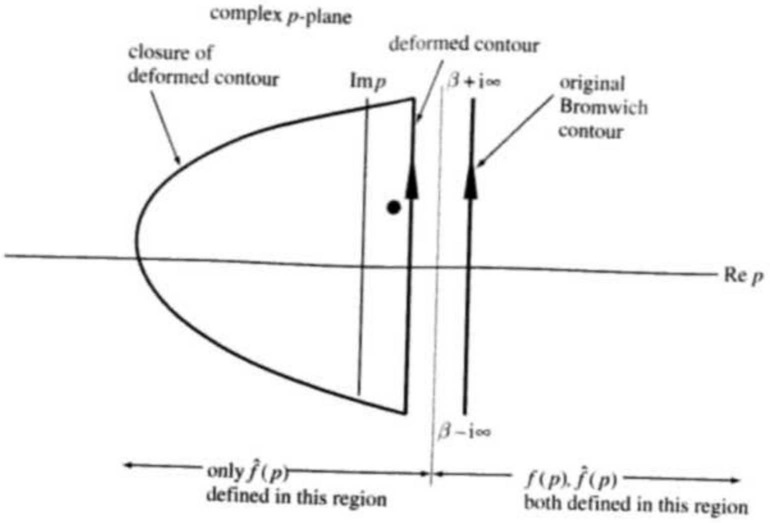

Fig. 5.4 Bromwich contour.

5.3.4 *Asymptotic long-time behavior of the potential oscillation*

We now return to the more daunting problem of evaluating Eq. (5.56). As in the simple example above, the goal is to close the contour to the left but, because the functions $N(p)$ and $D(p)$ are not defined for Re $p < \gamma$, this is not immediately possible. It is first necessary to construct analytic continuations of $N(p)$ and $D(p)$ that extend the definition of these functions into regions of negative Re p. As in the simple example, the desired analytic continuations may be constructed by taking the same formal expressions as obtained before, but now extending the definition to the entire p-plane with the proviso that *the functions remain analytic* as the region of definition is pushed leftwards in the p-plane.

Consider first construction of an analytic continuation for the function $N(p)$. This function can be written as

$$N(p) = \frac{1}{k^2\varepsilon_0}\sum_\sigma q_\sigma \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \frac{F_{\sigma 1}(v_\parallel, 0)}{(p+\mathrm{i}kv_\parallel)} = \frac{1}{\mathrm{i}k^3\varepsilon_0}\sum_\sigma q_\sigma \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \frac{F_{\sigma 1}(v_\parallel, 0)}{(v_\parallel - \mathrm{i}p/k)}. \quad (5.64)$$

Here, $\parallel$ means in the $\mathbf{k}$ direction, and the parallel component of the initial value of the perturbed distribution function has been defined as

$$F_{\sigma 1}(v_\parallel, 0) = \int \mathrm{d}^2\mathbf{v}_\perp f_{\sigma 1}(\mathbf{v}, 0). \quad (5.65)$$

The integrand in Eq. (5.54) has a pole at $v_\parallel = \mathrm{i}p/k$. Let us assume that $k > 0$ (the coordinate system can always be defined so that this is so). Before we construct an analytic continuation, Re p is restricted to be greater than γ so that the pole $v_\parallel = \mathrm{i}p/k$ is in the *upper half* of the complex $v_\parallel$-plane as shown in Fig. 5.5(a).

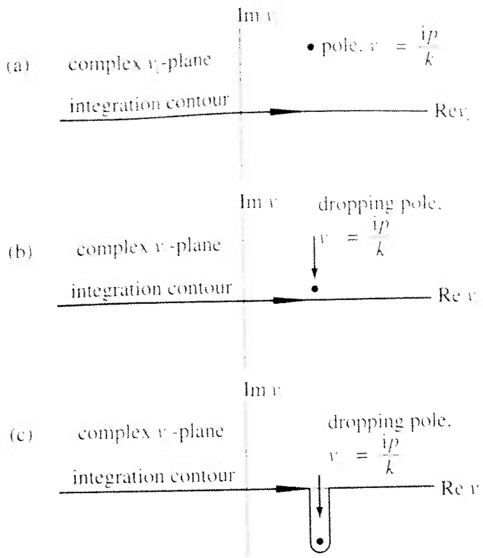

Fig. 5.5 Complex $v_\parallel$-plane.

When $N(p)$ is analytically continued to the left-hand region, the definition of $N(p)$ is extended to allow Re p to become less than γ and even negative. As shown in Fig. 5.5(b), decreasing Re p means that the pole at $v_\parallel = \mathrm{i}p/k$ in Eq. (5.54) drops from its initial location in the upper half $v_\parallel$-plane towards the lower half $v_\parallel$-plane. A critical question now arises: how should we arrange this construction when Re p passes through zero? If the pole is allowed to jump from being above the path of $v_\parallel$ integration (which is along the real $v_\parallel$ axis) to being below, the function $N(p)$ will *not* be analytic because it will have a discontinuous jump of $2\pi\mathrm{i}$ times the residue associated with the pole. Since it was stipulated that $N(p)$ must be analytic, the pole cannot be allowed to jump over the $v_\parallel$ contour of integration. Instead, the prescription proposed by Landau will be used, which is to *deform* the $v_\parallel$ contour as Re p becomes negative so that the contour *always* lies below the pole; this deformation is shown in Fig. 5.5(c).

$D(p)$ involves a similar integration along the real $v_\parallel$ axis. It also has a pole that is initially in the upper half-plane when Re $p > 0$, but then drops to being below the axis as Re p is allowed to become negative. Thus, analytic continuation of $D(p)$ is also constructed by deforming the path of the $v_\parallel$ integration so that the contour always lies below the pole.

Equipped with these suitably constructed analytic continuations of $N(p)$ and $D(p)$ into the left-hand p-plane, evaluation of Eq. (5.56) can now be undertaken. As shown in the simple example, it is computationally advantageous to deform the Bromwich contour into the left-hand p-plane. The deformed contour evaluates to the same result as the original Bromwich contour (provided the deformation does not jump over any poles) and this evaluation may be accomplished via the method of residues. In the general case where $N(p)/D(p)$ has several poles in the left-hand p-plane, then, as shown in Fig. 5.3(c), the contour may be deformed so that the vertical portion is pushed to the far left, except where there is a pole p_j; the contour "snags" around each pole p_j as shown in Fig. 5.3(c). For $\mathrm{Re}\, p \to -\infty$, the numerator $N(p) \to 0$, while the denominator $D(p) \to 1$. Since $\exp(pt) \to 0$ for $\mathrm{Re}\, p \to -\infty$ and positive t, the left-hand vertical line does not contribute to the integral and Eq. (5.56) simply consists of the sum of the residues of all the poles, i.e.,

$$\phi_1(t) = \sum_j \lim_{p \to p_j} \left[(p - p_j) \frac{N(p)}{D(p)} \mathrm{e}^{pt} \right]. \tag{5.66}$$

Where do the poles p_j come from? Upon examining Eq. (5.66), it is clear that poles could come either from (i) $N(p)$ having an explicit pole, i.e., $N(p)$ contains a term $\sim 1/(p - p_j)$, or (ii) from $D(p)$ containing a factor $\sim (p - p_j)$, i.e., p_j is a root of the equation $D(p) = 0$. The integrand in Eq. (5.64) has a pole in the $v_\parallel$-plane; this pole is "used up" as a residue upon performing the $v_\parallel$ integration, and so does not contribute a pole to $N(p)$. The only other possibility is that the initial value $F_{\sigma 1}(v_\parallel, 0)$ somehow provides a pole, but $F_{\sigma 1}(v_\parallel, 0)$ is a physical quantity with a bounded integral i.e., $\int F_{\sigma 1}(v_\parallel, 0) \mathrm{d}v_\parallel$ is finite and so cannot contribute a pole in $N(p)$. It is therefore concluded that all poles in $N(p)/D(p)$ must come from the roots (also called zeros) of $D(p)$.

The problem can be simplified by deciding to be content with a *less* than complete solution. Instead of attempting to calculate $\phi_1(t)$ for all positive times (i.e., all the poles p_j contribute to the solution), we restrict ourselves to the less burdensome problem of finding the long-time asymptotic behavior of $\phi_1(t)$. Because each term in Eq. (5.66) has a factor $\exp(\mathrm{i} p_j t)$, the least damped term (i.e., the term with pole furthest to the right in Fig. 5.3(c)) will dominate all the other terms at large t. Hence, in order to find the long-time asymptotic behavior, all that is required is to find the root p_j having the largest real part.

The problem is thus reduced to finding the roots of $D(p)$; this requires performing the $v_\parallel$ integration sketched in Fig. 5.5(c). Before doing this, it is convenient to integrate out the perpendicular velocity dependence from $D(p)$ so that

$$\begin{aligned} D(p) &= 1 - \frac{1}{k^2} \sum_\sigma \frac{q_\sigma^2}{\varepsilon_0 m_\sigma} \int \mathrm{d}^3 v \frac{\mathrm{i}\mathbf{k} \cdot \dfrac{\partial f_{\sigma 0}}{\partial \mathbf{v}}}{(p + \mathrm{i}\mathbf{k} \cdot \mathbf{v})} \\ &= 1 - \frac{1}{k^2} \sum_\sigma \frac{q_\sigma^2}{\varepsilon_0 m_\sigma} \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \frac{\dfrac{\partial F_{\sigma 0}}{\partial v_\parallel}}{(v_\parallel - \mathrm{i}p/k)}. \end{aligned} \tag{5.67}$$

Thus, the relation $D(p) = 0$ can be written in terms of susceptibilities as

$$D(p) = 1 + \chi_i + \chi_e = 0, \tag{5.68}$$

since the quantities being summed in Eq. (5.67) are essentially the electron and ion perturbations associated with the oscillation, and $D(p)$ is the Laplace transform analog of the Fourier transform of Poisson's equation. In the special case where the equilibrium distribution function is Maxwellian, the susceptibilities can be written in a standardized form as

$$\begin{aligned}
\chi_\sigma &= -\frac{1}{2k^2\lambda_{D\sigma}^2}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} \mathrm{d}\xi \frac{1}{(\xi - \mathrm{i}p/kv_{T\sigma})}\frac{\partial}{\partial \xi}\exp(-\xi^2) \\
&= \frac{1}{k^2\lambda_{D\sigma}^2}\left[\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} \mathrm{d}\xi \frac{(\xi - \mathrm{i}p/kv_{T\sigma} + \mathrm{i}p/kv_{T\sigma})}{(\xi - \mathrm{i}p/kv_{T\sigma})}\exp(-\xi^2)\right] \\
&= \frac{1}{k^2\lambda_{D\sigma}^2}\left[1 + \frac{1}{\pi^{1/2}}\alpha\int_{-\infty}^{\infty} \mathrm{d}\xi \frac{\exp(-\xi^2)}{(\xi - \alpha)}\right] \\
&= \frac{1}{k^2\lambda_{D\sigma}^2}\left[1 + \alpha Z(\alpha)\right],
\end{aligned} \tag{5.69}$$

where $\alpha = \mathrm{i}p/kv_{T\sigma}$, and the last line introduces the *plasma dispersion* function $Z(\alpha)$ defined as

$$Z(\alpha) \equiv \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} \mathrm{d}\xi \frac{\exp(-\xi^2)}{(\xi - \alpha)}, \tag{5.70}$$

where the ξ integration path is under the dropped pole.

5.3.5 Evaluation of the plasma dispersion function

If the pole corresponding to the fastest growing (i.e., least damped) mode turns out to have dropped well below the real axis (corresponding to Re p being large and negative), the fastest growing mode would be highly damped. We argue that this does not happen because there ought to be a correspondence between the Vlasov and fluid models in regimes where both are valid. Since the fluid model indicated the existence of undamped plasma waves when ω/k was much larger than the thermal velocity, the Vlasov model should predict nearly the same wave in this regime. The fluid wave model had no damping and so any damping introduced by the Vlasov model should be weak in order to maintain an approximate correspondence between fluid and Vlasov models. The Vlasov solution corresponding to the fluid mode can therefore have a pole only slightly below the real axis, i.e., only slightly negative. In this case, it is only necessary to analytically continue the definition of $N(p)/D(p)$ *slightly* into the negative p-plane. Thus, the pole in Eq. (5.70) drops only slightly below the real axis as shown in Fig. 5.6.

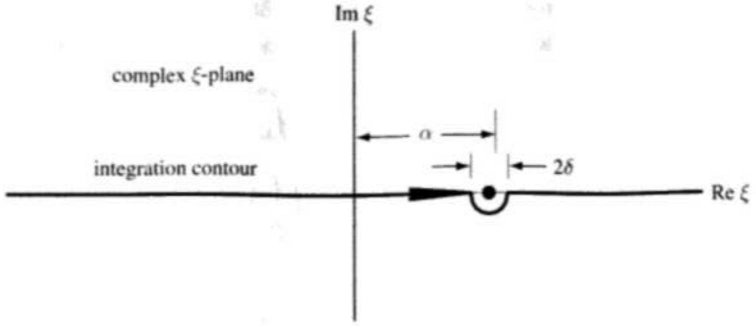

Fig. 5.6 Contour for evaluating plasma dispersion function.

The ξ integration contour can therefore be divided into three portions, namely (i) from $\xi = -\infty$ to $\xi = \alpha - \delta$, just to the left of the pole; (ii) a counterclockwise semi-circle of radius δ half-way around and *under* the pole (cf. Fig. 5.6); and (iii) a straight line from $\alpha + \delta$ to $+\infty$. The sum of the straight line segments (i) and (iii) in the limit $\delta \to 0$ is called the *principle part* of the integral and is denoted by a "P" in front of the integral sign. The semi-circle portion is *half* a residue and so makes a contribution that is just πi times the residue (rather than the standard 2πi for a complete residue). Hence, the plasma dispersion function for a pole slightly below the real axis is

$$Z(\alpha) = \frac{1}{\pi^{1/2}}\left[\mathrm{P}\int_{-\infty}^{\infty} d\xi \frac{\exp(-\xi^2)}{(\xi-\alpha)}\right] + i\pi^{1/2}\exp(-\alpha^2), \tag{5.71}$$

where P means principle part of the integral. Equation (5.71) prescribes how to evaluate ill-defined integrals of the type we first noted in Eq. (5.32).

There are two important limiting situations for $Z(\alpha)$, namely $|\alpha| \gg 1$ (corresponding to the adiabatic fluid limit since $\omega/k \gg v_{T\sigma}$) and $|\alpha| \ll 1$ (corresponding to the isothermal fluid limit since $\omega/k \ll v_{T\sigma}$). Asymptotic evaluations of $Z(\alpha)$ are possible in both cases and are found as follows:

1. $\alpha \gg 1$ case.
   Here, it is noted that the factor $\exp(-\xi^2)$ contributes significantly to the integral only when ξ is of order unity or smaller. In the important part of the integral where this exponential term is finite, $|\alpha| \gg \xi$. In this region of ξ the other factor in the integrand can be expanded as

$$\frac{1}{(\xi-\alpha)} = -\frac{1}{\alpha}\left(1-\frac{\xi}{\alpha}\right)^{-1} = -\frac{1}{\alpha}\left[1+\frac{\xi}{\alpha}+\left(\frac{\xi}{\alpha}\right)^2+\left(\frac{\xi}{\alpha}\right)^3+\left(\frac{\xi}{\alpha}\right)^4+\ldots\right]. \tag{5.72}$$

The expansion is carried to fourth order because of numerous cancelations that eliminate several of the lower order terms. Substitution of Eq. (5.72) into the integral in

Eq. (5.71) and noting that all odd terms in Eq. (5.72) do not contribute to the integral because the rest of the integrand is even gives

$$\mathrm{P}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}\xi\frac{\exp(-\xi^2)}{(\xi-\alpha)} = -\frac{1}{\alpha}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}\xi\exp(-\xi^2) \times\left[1+\left(\frac{\xi}{\alpha}\right)^2+\left(\frac{\xi}{\alpha}\right)^4+\ldots\right]. \tag{5.73}$$

The "P" has been dropped from the right-hand side of Eq. (5.73) because there is no longer any problem with a singularity. These Gaussian-type integrals may be evaluated by taking successive derivatives with respect to a of the Gaussian

$$\frac{1}{\pi^{1/2}}\int\mathrm{d}\xi\exp(-a\xi^2) = \frac{1}{a^{1/2}} \tag{5.74}$$

and then setting $a = 1$. Thus,

$$\frac{1}{\pi^{1/2}}\int\mathrm{d}\xi\,\xi^2\exp(-\xi^2) = \frac{1}{2}, \quad \frac{1}{\pi^{1/2}}\int\mathrm{d}\xi\,\xi^4\exp(-\xi^2) = \frac{3}{4} \tag{5.75}$$

so Eq. (5.73) becomes

$$\mathrm{P}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}\xi\frac{\exp(-\xi^2)}{(\xi-\alpha)} = -\frac{1}{\alpha}\left[1+\frac{1}{2\alpha^2}+\frac{3}{4\alpha^4}+\ldots\right]. \tag{5.76}$$

In summary, for $|\alpha| \gg 1$, the plasma dispersion function has the asymptotic form

$$Z(\alpha) = -\frac{1}{\alpha}\left[1+\frac{1}{2\alpha^2}+\frac{3}{4\alpha^4}+\ldots\right]+\mathrm{i}\pi^{1/2}\exp(-\alpha^2). \tag{5.77}$$

2. $|\alpha| \ll 1$ case.
   In order to evaluate the principle part integral in this regime, the variable $\eta = \xi - \alpha$ is introduced so that $\mathrm{d}\eta = \mathrm{d}\xi$. The integral may be evaluated as follows:

$$\begin{aligned}\mathrm{P}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}\xi\frac{\exp(-\xi^2)}{(\xi-\alpha)} &= \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}\eta\frac{\mathrm{e}^{-\eta^2-2\alpha\eta-\alpha^2}}{\eta}\\ &= \frac{\mathrm{e}^{-\alpha^2}}{\pi^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}\eta\frac{\mathrm{e}^{-\eta^2}}{\eta}\\ &\quad\times\left[1-2\alpha\eta+\frac{(-2\alpha)^2}{2!}+\frac{(-2\alpha)^3}{3!}+\ldots\right]\\ &= -2\alpha\frac{\mathrm{e}^{-\alpha^2}}{\pi^{1/2}}\int_{-\infty}^{\infty}\mathrm{d}\eta\,\mathrm{e}^{-\eta^2}\left[1+\frac{2\eta^2\alpha^2}{3}+\ldots\right]\\ &= -2\alpha\,(1-\alpha^2+\ldots)\left(1+\frac{\alpha^2}{3}+\ldots\right)\\ &= -2\alpha\left(1-\frac{2\alpha^2}{3}+\ldots\right).\end{aligned} \tag{5.78}$$

where in the third line all odd terms from the second line integrated to zero due to their symmetry. Thus, for $\alpha \ll 1$, the plasma dispersion function has the asymptotic limit

$$Z(\alpha) = -2\alpha\left(1 - \frac{2\alpha^2}{3} + \dots\right) + \mathrm{i}\pi^{1/2}\exp(-\alpha^2). \tag{5.79}$$

5.3.6 Landau damping of electron plasma waves

The plasma susceptibilities given by Eq. (5.69) can now be evaluated. For $|\alpha| \gg 1$, using Eq. (5.77), and introducing the "frequency" $\omega = \mathrm{i}p$ so that $\alpha = \omega/kv_{T\sigma}$ and $\alpha_i = \omega_i/kv_{T\sigma}$ the susceptibility is seen to be

$$\begin{aligned}
\chi_\sigma &= \frac{1}{k^2\lambda_{D\sigma}^2}\left\{1 + \alpha\left[-\frac{1}{\alpha}\left(1 + \frac{1}{2\alpha^2} + \frac{3}{4\alpha^4} + \dots\right) + \mathrm{i}\pi^{1/2}\exp(-\alpha^2)\right]\right\} \\
&= \frac{1}{k^2\lambda_{D\sigma}^2}\left\{-\left(\frac{1}{2\alpha^2} + \frac{3}{4\alpha^4} + \dots\right) + \mathrm{i}\alpha\pi^{1/2}\exp(-\alpha^2)\right\} \\
&= -\frac{\omega_{p\sigma}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{\kappa T_\sigma}{m_\sigma} + \dots\right) + \mathrm{i}\frac{\omega}{kv_{T\sigma}}\frac{\pi^{1/2}}{k^2\lambda_{D\sigma}^2}\exp(-\omega^2/k^2v_{T\sigma}^2).
\end{aligned} \tag{5.80}$$

Thus, if the root is such that $|\alpha| \gg 1$, the equation for the poles $D(p) = 1 + \chi_i + \chi_e = 0$ becomes

$$\begin{aligned}
1 &- \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{\kappa T_e}{m_e} + \dots\right) + \mathrm{i}\frac{\omega}{kv_{Te}}\frac{\pi^{1/2}}{k^2\lambda_{De}^2}\exp(-\omega^2/k^2v_{Te}^2) \\
&- \frac{\omega_{pi}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{\kappa T_i}{m_i} + \dots\right) + \mathrm{i}\frac{\omega}{kv_{Ti}}\frac{\pi^{1/2}}{k^2\lambda_{Di}^2}\exp(-\omega^2/k^2v_{Ti}^2) = 0.
\end{aligned} \tag{5.81}$$

This expression is similar to the previously obtained fluid dispersion relation, Eq. (4.32), but contains additional imaginary terms that did not exist in the fluid dispersion. Furthermore, Eq. (5.81) is not actually a dispersion relation. Instead, it is to be understood as the equation for the roots of $D(p)$. These roots determine the poles in $N(p)/D(p)$ producing the least damped oscillations resulting from some prescribed initial perturbation of the distribution function. Since $\omega_{pe}^2/\omega_{pi}^2 = m_i/m_e$, and in general $v_{Ti} \ll v_{Te}$, both the real and imaginary parts of the ion terms are much smaller than the corresponding electron terms. On dropping the ion terms, the expression becomes

$$1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{\kappa T_e}{m_e} + \dots\right) + \mathrm{i}\frac{\omega}{kv_{Te}}\frac{\pi^{1/2}}{k^2\lambda_{De}^2}\exp(-\omega^2/k^2v_{Te}^2) = 0. \tag{5.82}$$

Recalling that $\omega = \mathrm{i}p$ is complex, we write $\omega = \omega_r + \mathrm{i}\omega_i$ and then proceed to find the complex ω that is the root of Eq. (5.82). Although it would not be particularly difficult to substitute $\omega = \omega_r + \mathrm{i}\omega_i$ into Eq. (5.82) and then manipulate the coupled real and imaginary parts of this equation to solve for ω_r and ω_i, it is better to take this analysis as an opportunity to introduce a more general way for solving equations of this sort.

Equation (5.82) can be written as

$$D(\omega_r + \mathrm{i}\omega_i) = D_r(\omega_r + \mathrm{i}\omega_i) + \mathrm{i}D_i(\omega_r + \mathrm{i}\omega_i) = 0, \tag{5.83}$$

where D_r is the part of D that does not explicitly contain i and D_i is the part that does explicitly contain i. Thus,

$$D_r = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{\kappa T_e}{m_e} + \ldots\right), \qquad D_i = \frac{\omega}{k v_{Te}}\frac{\pi^{1/2}}{k^2\lambda_{De}^2}\exp\left(-\omega^2/k^2 v_{Te}^2\right). \tag{5.84}$$

Since the oscillation has been assumed to be weakly damped, $\omega_i \ll \omega_r$ and so Eq. (5.83) can be Taylor expanded in the small quantity ω_i,

$$D_r(\omega_r) + \mathrm{i}\omega_i\left(\frac{\mathrm{d}D_r}{\mathrm{d}\omega}\right)_{\omega=\omega_r} + \mathrm{i}\left[D_i(\omega_r) + \mathrm{i}\omega_i\left(\frac{\mathrm{d}D_i}{\mathrm{d}\omega}\right)_{\omega=\omega_r}\right] = 0. \tag{5.85}$$

Since $\omega_i \ll \omega_r$, the real part of Eq. (5.85) is

$$D_r(\omega_r) \simeq 0. \tag{5.86}$$

Balancing the two imaginary terms in Eq. (5.85) gives

$$\omega_i = -\frac{D_i(\omega_r)}{\dfrac{\mathrm{d}D_r}{\mathrm{d}\omega}}. \tag{5.87}$$

Thus, Eqs. (5.86) and (5.84) give the real part of the frequency as

$$\omega_r^2 = \omega_{pe}^2\left(1 + 3\frac{k^2}{\omega_r^2}\frac{\kappa T_e}{m_e}\right) \simeq \omega_{pe}^2\left(1 + 3k^2\lambda_{De}^2\right), \tag{5.88}$$

while Eqs. (5.87) and (5.84) give the imaginary part of the frequency, called the *Landau damping*, as

$$\begin{aligned}\omega_i &= -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{k^3\lambda_{De}^3}\exp\left(-\omega^2/k^2 v_{Te}^2\right)\\ &= -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{k^3\lambda_{De}^3}\exp\left[-\left(1 + 3k^2\lambda_{De}^2\right)/2k^2\lambda_{De}^2\right].\end{aligned} \tag{5.89}$$

Since the least damped oscillation goes as $\exp(pt) = \exp(-\mathrm{i}\omega t) = \exp(-\mathrm{i}(\omega_r + \mathrm{i}\omega_i)t) = \exp(-\mathrm{i}\omega_r t + \omega_i t)$ and Eq. (5.89) gives a negative ω_i, this is indeed a damping. It is interesting to note that while Landau damping was proposed theoretically by Landau in 1949, it took sixteen years before Landau damping was verified experimentally (Malmberg and Wharton 1964).

What is meant by weak damping vs. strong damping? In order to calculate ω_i it was assumed that ω_i is small compared to ω_r suggesting perhaps that ω_i is unimportant. However, even though small, ω_i can be important, because the factor 2π affects the real and imaginary parts of the wave phase differently. Suppose for example that the imaginary part of the frequency is $1/2\pi \sim 1/6$ the magnitude of the real part. This ratio is surely small enough to justify the Taylor expansion used in Eq. (5.85) and also to justify the assumption that the pole p_j corresponding to this mode is only slightly to the left of the imaginary p axis. Let us calculate how much the wave is attenuated in one period $\tau = 2\pi/\omega_r$. This attenuation will be $\exp(-|\omega_i|\tau) = \exp(-2\pi/6) \sim \exp(-1) \sim 0.3$. Thus, the wave amplitude decays to one third of its original value in just one period, which is certainly important.

5.3.7 Power relationships

It is premature to calculate the power associated with wave damping, because we do not yet know how to add up all the energy in the wave. Nevertheless, if we are willing to assume temporarily that the wave energy is entirely in the wave electric field (it turns out there is also energy in coherent particle motion – to be discussed in Chapter 14), it is seen that the power being lost from the wave electric field is

$$P_{wavelost} \sim \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\varepsilon_0 E_{wave}^2}{2} \right\rangle \sim \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\varepsilon_0 |E_{wave}^2|}{4} \exp\left(-2|\omega_i| t\right) \right] = -\frac{|\omega_i| \varepsilon_0 E_{wave}^2}{2}$$

$$= \sqrt{\frac{\pi}{8}} \frac{\omega_{pe}}{2k^3 \lambda_{De}^3} \exp\left(-\omega^2/k^2 v_{T\sigma}^2\right) \varepsilon_0 E_{wave}^2, \qquad (5.90)$$

where $\left\langle E_{wave}^2 \right\rangle = |E_{wave}|^2 \left\langle \cos(kx - \omega t) \right\rangle = |E_{wave}|^2/2$ has been used. However, in Section 3.8, it was shown that the energy gained by untrapped resonant particles in a wave is

$$P_{partgain} = \frac{-\pi m \omega}{2k^2} \left(\frac{q E_{wave}}{m} \right)^2 \left[\frac{\mathrm{d}}{\mathrm{d}v_0} f(v_0) \right]_{v_0 = \omega/k}$$

$$= \frac{-\pi m \omega}{2k^2} \left(\frac{q E_{wave}}{m} \right)^2 \left[\frac{\mathrm{d}}{\mathrm{d}v_0} \left\{ \left(\frac{m}{2\pi\kappa T} \right)^{1/2} n_0 \exp\left(-\frac{m v^2}{2\kappa T} \right) \right\} \right]_{v_0 = \omega/k}$$

$$= \frac{\pi m \omega}{2k^2} \left(\frac{q E_{wave}}{m} \right)^2 \left(\frac{m}{2\pi\kappa T} \right)^{1/2} \left(\frac{m}{\kappa T} \frac{\omega}{k} \right) n_0 \exp\left(-\frac{\omega^2}{k^2 v_{T\sigma}^2} \right); \qquad (5.91)$$

using $\omega \sim \omega_{pe}$ this is seen to be the same as Eq. (5.90) except for a factor of two. We shall see later that this factor of two comes from the fact that the wave